THE ALGEBRAIC GOODWILLIE TOWER

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1. The dual lambda algebra

We will use Singer's dual lambda algerba introduced in [6]

Let $GL_s = GL_s(\mathbb{F}_2)$ act on $\mathbb{F}_2[t_1, \dots, t_s]$ in the natural way. We have the subgroup $B_s \subset GL_s$ of upper triangular matrices. Then by [5] the B_s-invariants in $\mathbb{F}_2[t_1, \dots, t_s]$ in a polynomial algebra with generators

$$V_k = \prod(a_1 t_1 + \cdots + a_{k-1} t_{k-1} + t_k)$$

for $k = 1, \dots, s$, where the product is over all choices $a_i \in \mathbb{F}_2$.

Let $D_s = V_1 V_2 \dots V_s$ be the Dickson invariant, which is an invariant of GL_s. Then GL_s also acts on $\mathbb{F}_2[t_1, \dots, t_s, D^{-1}]$. We define Δ_s to be its B_s-invariants and Γ_s to be its GL_s-invariants. One finds $\Delta_s = \mathbb{F}_s[V_1^{\pm}, \dots, V_s^{\pm}]$.

Define another set of generators of Δ_s to be $v_k = \frac{V_k}{V_1 \dots V_{k-1}}$. Let Δ_s^- to be the span of the elements of the form $v_1^{i_1} \dots v_s^{i_s}$ in which at least one of the $i_j < 0$. Let $\Gamma_s^- = \Gamma_s \cap \Delta_s^-$, and $\Gamma_s^+ = \Gamma_s / \Gamma_s^-$.

Let $\Delta = \oplus \Delta_s$, $\Gamma = \oplus \Gamma_s$ and $\Gamma^+ = \oplus \Gamma_s^+$, etc.

Define $\partial_s : \Delta_s \to \Delta_{s-1}$ to be the map Res_{v_s}, which takes the coefficient before v_s^{-1}. It is proved in [6] that the restriction of ∂ to Γ makes Γ into a complex, and Γ^+ is a direct sumand of Γ as a complex.

There is a coproduct ψ on Δ defined by $\psi_{p,q} : \Delta_{p+q} \to \Delta_p \otimes \Delta_q$ where $\psi_{p,q}$ is an algebra isomorphism sending v_i to $v_i \otimes 1$ for $i \leq p$ and to $1 \otimes v_{i-p}$ for $i > p$. It is proved in [6] that Γ is a subcoalgebra of Δ and Γ^+ is a quotient coalgebra of Γ.

It is proved in [6] that Γ^+ is dual to the lambda algebra, with $v_1^{i_1} \dots v_s^{i_s}$ dual to $\lambda_{i_s} \dots \lambda_{i_1}$.

2. The Hecke algebra

We recall the basics of the mod 2 Hecke algebra in this section, following [2] and [3].

Let $\mathcal{H}_s$ be the mod 2 Hecke algebra. It has a set of basis $\{T_w\}$ for w running through the Weyl group W_s. $\mathcal{H}_s$ acts on the B_s-invariants in any GL_s-module such that T_w acts as $B_s w \in \mathbb{F}_2[GL_s]$. As an algebra, $\mathcal{H}_s$ is generated by T_{w_i} for $i = 1, \dots, s-1$, where w_i is the transposition of the i^{th} and $(i+1)^{st}$ element in the canonical basis of $\mathbb{F}_2^s$. We also define $e_s = T_{w_0}$ where w_0 inverts the order of the elements in the canonical basis. Let $\hat{e}_s = \sum_{w \in W_s} T_w$. Both e_s and $\hat{e}_s$ are idempotents mod 2. For any GL_s-module M, we can identify its GL_s invariants with $\hat{e}_s M^{B_s}$. We can also identify (or define) its Steinberg sumand to be $e_s M^{B_s}$.

Proposition 1. *The generators T_{w_i} satisfy the following relations mod 2, and these relations define $\mathcal{H}_s$:*

(1) $T_{w_i}^2 = T_{w_i}$.
(2) $T_{w_i}T_{w_{i+1}}T_{w_i} = T_{w_{i+1}}T_{w_i}T_{w_{i+1}}$.
(3) $T_{w_i}T_{w_j} = T_{w_j}T_{w_i}$ *for* $|i-j| \geq 2$.

From this proposition, we can define a map $\mathcal{H}_s \otimes \mathcal{H}_t \to \mathcal{H}_{s+t}$ by sending $T_{w_i} \otimes 1$ to T_{w_i} and $1 \otimes T_{w_i}$ to T_{w_s+1+i}. Define $e_t, \hat{e}_s \in \mathcal{H}_{s+t}$ to be the image of $1 \otimes e_t$ and $\hat{e}_s \otimes 1$ of this map respectively.

More generally, define $e_{i,j}$ to be the image of $1 \otimes e_{j+1-i} \otimes 1$ under the map $\mathcal{H}_{i-1} \otimes \mathcal{H}_{j-i+1} \otimes \mathcal{H}_{s-i-j} \to \mathcal{H}_s$

Proposition 2. *The elements e_i and $\hat{e}_i$ in $\mathcal{H}_{s+t}$ satisfy the following relations:*

(1) $e_t\hat{e}_s = \hat{e}_s e_t$.
(2) $e_i e_t = e_t = e_t e_i$ *for* $i \leq t$.
(3) $\hat{e}_i\hat{e}_s = \hat{e}_s = \hat{e}_s\hat{e}_i$ *for* $i \leq s$.
(4) $\hat{e}_s e_{t+1}\hat{e}_s + e_t\hat{e}_{s+1}e_t = \hat{e}_s e_t$.

Define $\hat{T}_{w_i} = 1 + T_{w_i}$.

Proposition 3. *In $\mathcal{H}_s$, we have*

(1) $\hat{T}_{w_i}\hat{e}_s = \hat{e}_s = \hat{e}_s\hat{T}_{w_i}$.
(2) $\hat{e}_2 = \hat{T}_{w_1}$.
(3) $\hat{e}_k = \hat{T}_{w_1}\dots\hat{T}_{w_{k-1}}\hat{e}_{k-1}$.

In the following we will abbreviate T_{w_i} by T_i, etc.

Now specialize to the case Δ_s. $\mathcal{H}_s$ acts on Δ_s, and one can identify Γ_s as $\hat{e}_s\Delta_s$. The map $\psi_{p,q}$ preserves the action of $\mathcal{H}_p \otimes \mathcal{H}_q$. The map ∂_s preserves the action of $\mathcal{H}_{s-1}$.

3. The complexes $\mathcal{L}(n)$

In this section, we define complexes $\mathcal{L}(n)$ which are isomorphic to the dual of Λ-modules of the spectra $L(n)$.

Define $\mathcal{L}(n)_s = \hat{e}_{s-n}e_n\Delta_s$ for $s \geq n$ and $\mathcal{L}(n)_s = 0$ for $s < n$. Define the differential by the formula $\partial'_s : \mathcal{L}(n)_s \to \mathcal{L}(n)_{s-1}$ to be

$$\partial_s(x)' = Res_{v_s}(\hat{T}_{s-1}\dots\hat{T}_{s-n}x)$$

for any $x \in \mathcal{L}(n)_s \subset \Delta_s$.

Proposition 4. *The map ∂' lands in $\mathcal{L}(n)$.*

Proof. It is trivial to check $\hat{e}_{s-n-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x = \hat{T}_{s-1}\dots\hat{T}_{s-n}x$. To complete the proof, it remains to check the equations $\hat{T}_k\hat{T}_{s-1}\dots\hat{T}_{s-n}x = 0$ for $k = s-n,\dots,s-2$. This can be done with the commutation relations of the T_i's. First we can transform the expression by moving T_k rightward to arrive at the expression $\hat{T}_{s-1}\dots\hat{T}_k\hat{T}_{k+1}\hat{T}_k\dots\hat{T}_{s-n}x$. Using the braid relation, this equals $\hat{T}_{s-1}\dots\hat{T}_{k+1}\hat{T}_k\hat{T}_{k+1}\dots\hat{T}_{s-n}x$. Then we move the right $\hat{T}_{k+1}$ further rightward, and end with $\hat{T}_{s-1}\dots\hat{T}_{k+1}\hat{T}_k\dots\hat{T}_{s-n}\hat{T}_{k+1}x$. Finally observe $\hat{T}_{k+1}x = 0$ for $k = s-n,\dots,s-1$. □

Define $\mathcal{L}(n)_s^+ = \mathcal{L}(n)_s/\mathcal{L}(n)_s \cap \Delta^-$.

Theorem 1. *The map ∂' satisfies $\partial'^2 = 0$. Moreover, $\mathcal{L}(n)^+$ is isomorphic to the dual of Λ-modules of the spectra $L(n)$.*

Proof. The first assertion is a consequence of Proposition 3.1 in [6], which implies ∂_s^2 restricted to the image of $\hat{T}_{s-1}$ is zero, provided we can show any expression $\hat{T}_{s-2}\dots\hat{T}_{s-n-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x$ can be transformed into an expression starting with $\hat{T}_{s-1}$. To do this, first move $\hat{T}_{s-n-1}$ rightward to arrive at the expression $\hat{T}_{s-2}\dots\hat{T}_{s-n}\hat{T}_{s-1}\dots\hat{T}_{s-n-1}\hat{T}_{s-n}x$. We know $x = \hat{T}_{s-n-1}x$. So $\hat{T}_{s-n-1}\hat{T}_{s-n}x = \hat{T}_{s-n}\hat{T}_{s-n-1}\hat{T}_{s-n}x$. So we get an expression starting with $\hat{T}_{s-2}\dots\hat{T}_{s-n}\hat{T}_{s-1}\dots\hat{T}_{s-n+1}\hat{T}_{s-n}$. Then an induction completes the proof.

The second assertion can be proved by comparing with the formula of the action of the Steenrod algebra on $H^*(L(n))$ described in [1] using the Nishida relations. □

4. The Kuhn-Priddy map and the transfer map

We define the Kuhn-Priddy map $s_n : \mathcal{L}(n+1) \to \mathcal{L}(n)$ by the formula $s_n(x) = \hat{e}_{s-n}x$ for $x \in \mathcal{L}(n+1)_s$.

Proposition 5. *The map s_n is a map of complexes.*

Proof. Let $x \in \mathcal{L}(n+1)_s$. Then

$$\partial'(s_n(x)) = Res_{v_s}\hat{T}_{s-1}\dots\hat{T}_{s-n}\hat{e}_{s-n}x$$

We have $x = \hat{e}_{s-n-1}x$, and $\hat{e}_{s-n} = \hat{e}_{s-n-1}T_{s-n-1}\hat{e}_{s-n-1}$, so

$$\hat{T}_{w_{s-1}}\dots\hat{T}_{w_{s-n}}\hat{e}_{s-n}x = \hat{e}_{s-n-1}\hat{T}_{s-1}\dots\hat{T}_{s-n-1}x$$

because $\hat{e}_{s-n-1}$ commutes with $\hat{T}_{s-n},\dots,\hat{T}_{s-1}$.

On the other hand, we have

$$s_n(\partial'x) = Res_{v_s}\hat{e}_{s-n-1}\hat{T}_{s-1}\dots\hat{T}_{s-n-1}x$$

□

Similarly, we define the transfer map $d_n : \mathcal{L}(n) \to \mathcal{L}(n+1)$ by the formula $d_n(x) = e_{n+1}x$ for $x \in \mathcal{L}(n+1)_s$.

Proposition 6. *The map d_n is a map of complexes.*

Proof. Let $x \in \mathcal{L}(n)_s$. Then

$$\partial'(d_n(x)) = Res_{v_s}\hat{T}_{s-1}\dots\hat{T}_{s-n-1}e_{n+1}x$$

We have $e_{n+1} = e_n + e_n\hat{T}_{s-n}e_n$, and $e_nx = x = \hat{T}_{s-n-1}x$. So

$$\hat{T}_{s-1}\dots\hat{T}_{s-n-1}e_{n+1}x = \hat{T}_{s-1}\dots\hat{T}_{s-n}x + \hat{T}_{s-1}\dots\hat{T}_{s-n-1}e_n\hat{T}_{s-n}x$$

Because e_n commutes with $\hat{T}_{s-n-1}$, we have

$$\hat{T}_{s-1}\dots\hat{T}_{s-n-1}e_n\hat{T}_{s-n}x = \hat{T}_{s-1}\dots\hat{T}_{s-n}e_n\hat{T}_{s-n-1}\hat{T}_{s-n}x$$

On the other hand, we have

$$d_n(\partial'x) = Res_{v_s}e_{s-n-1,s-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x$$

We have

$$e_{s-n-1,s-1} = e_{s-n,s-1} + e_{s-n,s-1}\hat{T}_{s-n-1}e_{s-n,s-1}$$

From the proof of Theorem 1, we know

$$e_{s-n,s-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x=\hat{T}_{s-1}\dots\hat{T}_{s-n}x$$

So

$$e_{s-n-1,s-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x=\hat{T}_{s-1}\dots\hat{T}_{s-n}x+e_{s-n,s-1}\hat{T}_{s-n-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x$$

We also have

$$e_{s-n,s-1}\hat{T}_{s-n-1}\hat{T}_{s-1}\dots\hat{T}_{s-n}x=e_{s-n,s-1}\hat{T}_{s-1}\dots\hat{T}_{s-n+1}\hat{T}_{s-n-1}\hat{T}_{s-n}x$$

Then an induction on n proves the proposition once we notice

$$e_{n-1}\hat{T}_{s-n-1}\hat{T}_{s-n}x=\hat{T}_{s-n-1}\hat{T}_{s-n}x$$

□

The following is a direct consequence of the formula

$$\hat{e}_s e_{t+1}\hat{e}_s+e_t\hat{e}_{s+1}e_t=\hat{e}_s e_t$$

Proposition 7. *We have $d_n s_n+s_{n-1}d_{n-1}=1$.*

This is consistent with the algebraic Whitehead conjecture proved in [4].

References

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